9.

Integration techniques and applications

- Integration by substitution (or change of variable)
- Definite integrals and the change of variable method
- Use of trigonometric identities to assist integration
- Integration to give logarithmic functions
- More about partial fractions
- Volumes of revolution
- Rotation about the *y*-axis
- Numerical integration
- Extension: Integration by parts
- Miscellaneous exercise nine

Integration by substitution (or change of variable)

Integration by substitution

Asked to determine $\int 48x(3x^2 - 1)^3\, dx$, and noticing that $48x(3x^2 - 1)^3$ is of the form $f'(x)\,[f(x)]^n$, except for some scalar multiple, we could try

$$\begin{aligned}
y &= (3x^2 - 1)^4 \\
\text{in which case} \quad \frac{dy}{dx} &= (4)(3x^2 - 1)^3(6x) \\
&= 24x\,(3x^2 - 1)^3 \\
\text{Thus} \quad \int 48x(3x^2 - 1)^3\, dx &= 2(3x^2 - 1)^4 + c.
\end{aligned}$$

Alternatively we could make a suitable **substitution**, in this case $u = 3x^2 - 1$, to **change the variable** involved from x to u, as follows.

$$\text{If} \quad u = 3x^2 - 1 \quad \text{then} \quad \frac{du}{dx} = 6x$$

$$\begin{aligned}
\text{Thus} \quad \int 48x(3x^2 - 1)^3\, dx &= \int 48x(3x^2 - 1)^3 \frac{dx}{du}\, du \\
&= \int 48xu^3 \frac{1}{6x}\, du \\
&= \int 8u^3\, du \\
&= 2u^4 + c \\
&= 2(3x^2 - 1)^4 + c \qquad \text{as before.}
\end{aligned}$$

We do not express the $48x$ in terms of u because we can see the $48x$ and the $6x$ cancelling.

In the example above, this method of substitution, or change of variable, holds no great advantage over our initial 'trial and adjustment' method, provided of course that we do notice the suitability of trialling $(3x^2 - 1)^4$. However we will meet situations for which an initial sensible trial may not be at all obvious but making a suitable substitution does yield a solution. The next two examples are of this type.

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EXAMPLE 1

Use the substitution $u = 2x - 3$ to determine $\int 56x(2x-3)^5\, dx$.

Solution

If $\quad u = 2x - 3 \quad$ then $\quad \dfrac{du}{dx} = 2$

Thus

$$\begin{aligned}
\int 56x(2x-3)^5\, dx &= \int 56x(2x-3)^5 \frac{dx}{du}\, du \\
&= \int 56\left(\frac{u+3}{2}\right)u^5 \frac{1}{2}\, du \\
&= \int (14u^6 + 42u^5)\, du \\
&= 2u^7 + 7u^6 + c \\
&= u^6(2u+7) + c \\
&= (2x-3)^6(4x+1) + c
\end{aligned}$$

EXAMPLE 2

Use the substitution $u = x - 2$ to determine $\int \dfrac{6x}{\sqrt{x-2}}\, dx$.

Solution

If $\quad u = x - 2 \quad$ then $\quad \dfrac{du}{dx} = 1.$

Thus

$$\begin{aligned}
\int \frac{6x}{\sqrt{x-2}}\, dx &= \int \frac{6x}{\sqrt{x-2}} \frac{dx}{du}\, du \\
&= \int \frac{6(u+2)}{\sqrt{u}}(1)\, du \\
&= \int (6u^{\frac{1}{2}} + 12u^{-\frac{1}{2}})\, du \\
&= 4u^{\frac{3}{2}} + 24u^{\frac{1}{2}} + c \\
&= 4u^{\frac{1}{2}}(u+6) + c \\
&= 4\sqrt{x-2}\,(x+4) + c
\end{aligned}$$

$\int \left(\dfrac{6\cdot x}{\sqrt{x-2}}\right) dx$

$4\cdot\sqrt{x-2}\cdot(x+4)$

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Note
- In the previous examples, the final answers are given in terms of the variable given in the question (x in the previous examples), not in terms of the variable introduced to aid the integration (u in the previous examples).
- The replacement of 'dx' by $\frac{dx}{du}du$ in these examples can be thought of as being reasonable by thinking of the 'du's cancelling. However this is not really the case because $\frac{dx}{du}$ is not a fraction, it is the limit of a fraction.

  A more formal proof of the fact that if $f(x) = g(u)$ then

$$\int f(x)\,dx = \int g(u)\frac{dx}{du}\,du$$

  is not included here.

Exercise 9A

Determine the following integrals using the suggested substitution.

1 $\int 60x(x^2 - 3)^5\,dx$, $u = x^2 - 3$

2 $\int 80x(1 - 2x)^3\,dx$, $u = 1 - 2x$

3 $\int 12x(3x + 1)^5\,dx$, $u = 3x + 1$

4 $\int 6x(2x^2 - 1)^5\,dx$, $u = 2x^2 - 1$

5 $\int 12x(3x^2 + 1)^5\,dx$, $u = 3x^2 + 1$

6 $\int 3x(x - 2)^5\,dx$, $u = x - 2$

7 $\int 20x(3 - x)^3\,dx$, $u = 3 - x$

8 $\int 4x(5 - 2x)^5\,dx$, $u = 5 - 2x$

9 $\int 20x(2x + 3)^3\,dx$, $u = 2x + 3$

10 $\int 18x\sqrt{3x + 1}\,dx$, $u = 3x + 1$

11 $\int \frac{6x}{\sqrt{3x^2 + 5}}\,dx$, $u = 3x^2 + 5$

12 $\int \frac{3x}{\sqrt{1 - 2x}}\,dx$, $u = 1 - 2x$

13 $\int 8\sin^5 2x\cos 2x\,dx$, $u = \sin 2x$

14 $\int 27\cos^7 3x\sin 3x\,dx$, $u = \cos 3x$

15 $\int 6x\sin(x^2 + 4)\,dx$, $u = x^2 + 4$

16 $\int (4x + 3)(2x + 1)^5\,dx$, $u = 2x + 1$

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Exercise 9B

Determine the following integrals using any appropriate method.

1 $\int (x + \sin 3x)\, dx$

2 $\int 2\, dx$

3 $\int \sin 8x\, dx$

4 $\int (\cos x + \sin x)(\cos x - \sin x)\, dx$

5 $\int \frac{x^2 + x}{\sqrt{x}}\, dx$

6 $\int 4x \sin(x^2)\, dx$

7 $\int 8x \sin(x^2 - 3)\, dx$

8 $\int 24\sqrt{1+3x}\, dx$

9 $\int 15x\sqrt{1+3x}\, dx$

10 $\int \sin^4 2x \cos 2x\, dx$

11 $\int 6x(2x+7)^5\, dx$

12 $\int 6(2x+7)^5\, dx$

13 $\int (3x^2 - 2)\, dx$

14 $\int 4x(3x^2 - 2)^7\, dx$

15 $\int (\cos x + \sin 2x)\, dx$

16 $\int 6x(3x-2)^7\, dx$

17 $\int x\, dx$

18 $\int \frac{6}{\sqrt{1+2x}}\, dx$

19 $\int \frac{6x}{\sqrt{1+2x}}\, dx$

20 $\int (x^2 + x + 1)^8 (2x+1)\, dx$

21 $\int 24x \sin(x^2 + 3)\, dx$

22 $\int (2x+1)\sqrt[3]{x-5}\, dx$

23 $\int \frac{(\sqrt{x}+5)^5}{\sqrt{x}}\, dx$

24 $\int 4(2x-1)^5\, dx$

25 $\int 4x(2x-1)^5\, dx$

26 $\int \cos^3 6x \sin 6x\, dx$

27 $\int \frac{6x}{\sqrt{x^2 - 3}}\, dx$

28 $\int \sin 2x \cos 2x\, dx$

29 $\int 8x^2(2x-1)^5\, dx$

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Definite integrals and the change of variable method

If the change of variable method is used to evaluate definite integrals, care needs to be taken with the upper and lower limits, as shown in the next example.

EXAMPLE 3

Use the substitution $u = 2x + 1$ to determine $\int_0^4 \frac{8x}{\sqrt{2x+1}}\,dx$.

Solution

If $u = 2x + 1$ then $\frac{du}{dx} = 2$.

Thus
$$\begin{aligned}
\int_0^4 \frac{8x}{\sqrt{2x+1}}\,dx &= \int_{x=0}^{x=4} \frac{8x}{\sqrt{2x+1}}\frac{dx}{du}\,du \\
&= \int_{u=1}^{u=9} \frac{4(u-1)}{\sqrt{u}}\frac{1}{2}\,du \\
&= \int_1^9 (2u^{\frac{1}{2}} - 2u^{-\frac{1}{2}})\,du \\
&= \left[\frac{4u^{\frac{3}{2}}}{3} - 4u^{\frac{1}{2}}\right]_1^9 \\
&= 26\frac{2}{3}
\end{aligned}$$

$$\int_0^4 \frac{8x}{\sqrt{2x+1}}\,dx$$

$$\frac{80}{3}$$

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Exercise 9C

Determine the following definite integrals using the suggested substitution and showing full algebraic reasoning. Confirm each answer using a graphic calculator.

1 $\int_0^1 16(2x+1)^3 dx,$ $\quad u = 2x + 1$

2 $\int_0^1 16x(2x+1)^3 dx,$ $\quad u = 2x + 1$

3 $\int_0^1 \frac{6x}{25}(x+5)^4\, dx,$ $\quad u = x + 5$

4 $\int_0^{\frac{\pi}{2}} 12\sin^5 x \cos x\, dx,$ $\quad u = \sin x$

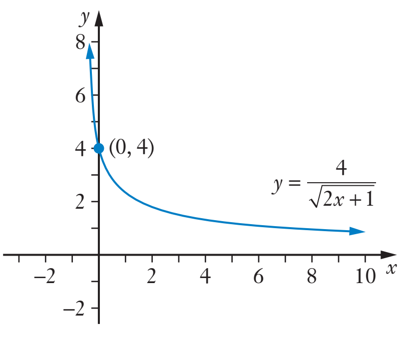
5 $\int_2^6 \frac{3x}{\sqrt{5x+6}}\, dx,$ $\quad u = 5x + 6$

6 $\int_2^5 \frac{x+3}{\sqrt{x-1}}\, dx,$ $\quad u = x - 1$

7 The graph on the right is that of the function

$$y = \frac{4}{\sqrt{2x+1}}.$$

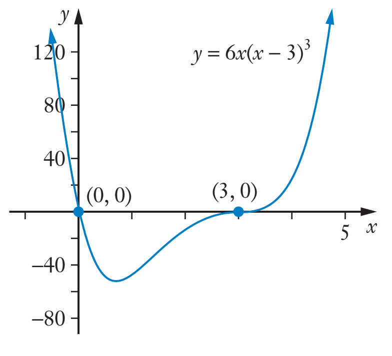
Performing any integration using the substitution $u = 2x + 1$, determine the area under the curve from $x = 0$ to $x = 4$.



8 The graph on the right is that of the function

$$y = 6x(x-3)^3.$$

Performing any integration using the substitution $u = x - 3$, determine the area enclosed by the curve and the x-axis.



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Use of trigonometric identities to assist integration

WS Trigonometric identities

Some integrations are best performed by first rearranging the function to be integrated using some of the trigonometric identities we were reminded of in the *Preliminary work* section at the beginning of this unit. The examples that follow demonstrate such use.

In particular, note carefully the techniques for finding the antiderivatives of

$$\sin^n x \text{ and } \cos^n x$$

as demonstrated in example 4, for **when *n* is odd**, and the technique ensures that some terms of the form $f'(x)\,[f(x)]^k$ arise,

and in example 5, for **when *n* is even**, and the technique uses the trigonometric identity $\cos 2x = 2\cos^2 x - 1$,

rearranged as $\cos^2 x = \dfrac{1+\cos 2x}{2}$.

EXAMPLE 4

Find the antiderivative of $\sin^5 x$.

Solution

$$\begin{aligned}\sin^5 x &= (\sin x)(\sin^4 x)\\ &= (\sin x)(1-\cos^2 x)^2\\ &= (\sin x)(1-2\cos^2 x+\cos^4 x)\\ &= \sin x - 2\sin x\cos^2 x + \sin x\cos^4 x\end{aligned}$$

To find the antiderivative, try $y = \cos x + \cos^3 x + \cos^5 x$

then $\dfrac{dy}{dx} = -\sin x - 3\cos^2 x\sin x - 5\cos^4 x\sin x$

Thus if $y = -\cos x + \dfrac{2}{3}\cos^3 x - \dfrac{1}{5}\cos^5 x$

then, as required, $\dfrac{dy}{dx} = \sin x - 2\sin x\cos^2 x + \sin x\cos^4 x.$

The required antiderivative is $-\cos x + \dfrac{2}{3}\cos^3 x - \dfrac{1}{5}\cos^5 x + c.$

Does your calculator give this answer when asked for $\int \sin^5 x\, dx$?

If it gives an answer that looks different to that shown above are they in fact equivalent? Investigate.

EXAMPLE 5

Find the antiderivative of $\cos^4 x$.

Solution

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2 \\ &= \left(\frac{1+\cos 2x}{2}\right)^2 \\ &= \frac{1}{4} + \frac{2\cos 2x}{4} + \frac{\cos^2 2x}{4} \\ &= \frac{1}{4} + \frac{\cos 2x}{2} + \frac{1+\cos 4x}{8} \\ &= \frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}\end{aligned}$$

The required antiderivative is $\frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$.

Again check to see if this is what your calculator gives when asked for $\int \cos^4 x\, dx$.

EXAMPLE 6

Use the fact that $$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

to determine $$\int \sin 5x \cos 3x\, dx$$

Solution

$$\begin{aligned}\sin 5x \cos 3x &= \frac{1}{2}[\sin(5x+3x) + \sin(5x-3x)] \\ &= \frac{1}{2}\sin(8x) + \frac{1}{2}\sin(2x)\end{aligned}$$

Thus
$$\begin{aligned}\int \sin 5x \cos 3x\, dx &= \int \frac{1}{2}\sin(8x)\, dx + \int \frac{1}{2}\sin(2x)\, dx \\ &= -\frac{1}{16}\cos(8x) - \frac{1}{4}\cos(2x) + c\end{aligned}$$

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Before proceeding to the next example recall first the derivative of $\tan x$:

$$\begin{aligned} y &= \tan x \\ &= \frac{\sin x}{\cos x} \end{aligned}$$

By the quotient rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x\,. \end{aligned}$$

Thus $$\frac{d}{dx}(\tan x) = \sec^2 x\,.$$

EXAMPLE 7

Determine $$\int \tan^2 x\,dx.$$

Solution

The technique with this integration is to use the identity $\tan^2 x + 1 = \sec^2 x$

$$\begin{aligned} \int \tan^2 x\,dx &= \int (\sec^2 x - 1)\,dx \\ &= \int \sec^2 x\,dx - \int 1\,dx \\ &= \tan x - x + c \end{aligned}$$

Exercise 9D

1 Use the fact that $$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

to determine $$\int \cos 5x \cos 4x\,dx.$$

2 Use the fact that $$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

to determine $$\int \sin 7x \sin x\,dx.$$

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Find the antiderivative of each of the following.
(Note: Not all of the expressions need rearranging.)

3 $\sin^4 x \cos x$

4 $6\sin^3 x \cos x$

5 $\sin^3 x$

6 $\cos^3 x$

7 $\cos^5 x$

8 $\cos^2 x$

9 $\sin^2 x$ (Hint: $\cos 2A = 1 - 2\sin^2 A$.)

10 $8\sin^4 x$

11 $\cos^2 x + \sin^2 x$

12 $\cos^2 x - \sin^2 x$

13 $\sin^3 x + \cos^2 x$

14 $2\sin x \cos x$

15 $\sin^3 x \cos^2 x$

16 $\cos^3 x \sin^2 x$

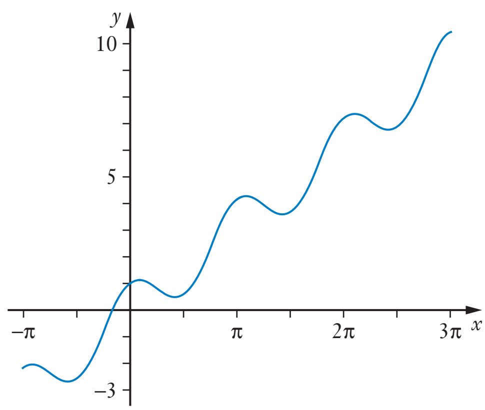
17 $\tan^2 3x$

18 $1 + \tan^2 x$

19 $\dfrac{\sin x}{1-\sin x} \times \dfrac{\sin x}{1+\sin x}$

20 $\sec^2 x \tan^4 x$

21 The graph shown below shows the function $y = x + \cos^2 x - \sin^2 x$, for $-\pi \le x \le 3\pi$.



Determine the area under the curve $y = x + \cos^2 x - \sin^2 x$ from $x = 0$ to $x = 2\pi$.

22 A particle moves such that its velocity vector, $\mathbf{v}$ m/s, at time t seconds is given by

$$\mathbf{v} \quad = \quad 4\sin^2 t\,\mathbf{i} + \tan^2 t\,\mathbf{j} \qquad (0 \le t \le \frac{\pi}{2}).$$

Find **a** an expression for the position vector of the particle, $\mathbf{r}(t)$ metres, at time t seconds given that when $t = 0$, $\mathbf{r} = 3\mathbf{i} + \mathbf{j}$.

b the position vector of the particle when $t = \dfrac{\pi}{4}$.

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Integration to give logarithmic functions

WS Integration of $\frac{1}{x}$

It is assumed that from your study of *Mathematics Methods* Unit Four you are now familiar with the idea of logarithmic functions. In particular you will have encountered, or soon will encounter, the fact that:

Any algebraic fraction for which the numerator is the derivative of the denominator will integrate to give a natural logarithmic function.

$$\int \frac{1}{x}\,dx = \ln x + c. \qquad \int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c.$$

In the real number system the logarithmic function $\ln x$ is only defined for $x > 0$. Thus the integral above left only has meaning for $x > 0$ and above right only for $f(x) > 0$. In the treatment of logarithmic functions in *Mathematics Methods* Unit Four, consideration of integrals of the form

$$\int \frac{f'(x)}{f(x)}\,dx$$

are restricted to $f(x) > 0$. However, the companion text for that unit, does say:

Suppose we were asked to determine $\int \frac{1}{x}\,dx$ for $x < 0$.

Writing the answer as $\ln x + c$ would present a problem because we would then be faced with the logarithm of a negative number.

However, this situation is avoidable if, for $x < 0$, we were to write $\int \frac{1}{x}\,dx$ as $\int \frac{-1}{-x}\,dx$, for which the answer is $\ln(-x) + c$.

Thus we could say that for $x > 0$, $\int \frac{1}{x}\,dx = \ln x + c$,

and for $x < 0$, $\int \frac{1}{x}\,dx = \int \frac{-1}{-x}\,dx = \ln(-x) + c$.

Combining these two statements using the absolute value gives

$$\int \frac{1}{x}\,dx = \ln|x| + c. \qquad x \neq 0.$$

- In *Mathematics Methods* Unit Four this point was mentioned to explain why your calculator may, when asked to determine $\int \frac{1}{x}\,dx$, display an answer that includes the absolute value.
- In this *Mathematics Specialist* unit we will use the more general results:

$$\int \frac{1}{x}\,dx = \ln|x| + c. \qquad x \neq 0.$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c. \qquad f(x) \neq 0.$$

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EXAMPLE 8

Find each of the following integrals.

a $\displaystyle\int \frac{14}{2x-1}\,dx$ **b** $\displaystyle\int \frac{6x}{x^2+1}\,dx$

Solution

a Noticing that the numerator is a scalar multiple of the derivative of the denominator, we expect the answer to involve $\ln|2x-1|$.

$$\int \frac{14}{2x-1}\,dx = 7\ln|2x-1| + c. \qquad \text{(See first note below.)}$$

b Noticing that the numerator is a scalar multiple of the derivative of the denominator we expect the answer to involve $\ln\left|x^2+1\right|$.

$$\int \frac{6x}{x^2+1}\,dx = 3\ln\left|x^2+1\right| + c. \qquad \text{(See second note below.)}$$

Note
- Using logarithmic laws we could write the answer to part **a** as follows:

$$\begin{aligned} 7\ln|2x-1| + c &= 7\ln(2|x-0.5|) + c \\ &= 7\ln 2 + 7\ln|x-0.5| + c \\ &= 7\ln|x-0.5| + \text{a constant.} \end{aligned}$$

- With (x^2+1) positive for all x the answer to part **b** could be written simply as $3\ln(x^2+1)+c$.

EXAMPLE 9

Find each of the following definite integrals.

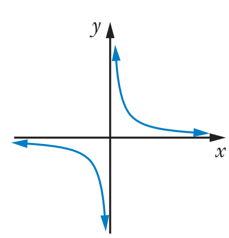
a $\displaystyle\int_2^3 \frac{1}{x}\,dx$ **b** $\displaystyle\int_{-3}^{-2} \frac{1}{x}\,dx.$

Solution

a
$$\begin{aligned} \int_2^3 \frac{1}{x}\,dx &= \Big[\ln|x|\Big]_2^3 \\ &= \ln 3 - \ln 2 \\ &= \ln 1.5 \end{aligned}$$

b
$$\begin{aligned} \int_{-3}^{-2} \frac{1}{x}\,dx &= \Big[\ln|x|\Big]_{-3}^{-2} \\ &= \ln 2 - \ln 3 \\ &= -\ln 1.5 \end{aligned}$$

Note
- That the part **b** answer above is the negative of the part **a** answer should come as no surprise when we consider the graph of $y = \dfrac{1}{x}$.



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- Definite integrals of the form $\int_a^b \frac{f'(x)}{f(x)}\,dx$ are undefined if, for some value of x in the interval $a \le x \le b$, $f(x) = 0$.

  For example: $\int_{-1}^{2} \frac{1}{x}\,dx$ is undefined. $\int_2^5 \frac{1}{(x^2 - 2x - 3)}\,dx$ is undefined.

  How does your calculator respond when asked to find these definite integrals?

EXAMPLE 10

Find each of the following integrals.

a $\int \frac{2x}{x+1}\,dx$ **b** $\int \frac{2x+5}{x(x+1)}\,dx$

Solution

a First rearrange the improper fraction:

$$\frac{2x}{x+1} = \frac{2(x+1)-2}{x+1} = 2 - \frac{2}{x+1}$$

Thus $$\int \frac{2x}{x+1}\,dx = \int 2\,dx - \int \frac{2}{x+1}\,dx = 2x - 2\ln|x+1| + c.$$

b First express $\frac{2x+5}{x(x+1)}$ in what we call **partial fractions**, as shown below.

We write $$\frac{2x+5}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)}$$

Hence $2x + 5 = A(x+1) + Bx$

from which $2 = A + B$ and $5 = A$.

Thus $A = 5$ and $B = -3$.

$$\int \frac{2x+5}{x(x+1)}\,dx = \int \frac{5}{x}\,dx - \int \frac{3}{x+1}\,dx = 5\ln|x| - 3\ln|x+1| + c.$$

More about partial fractions

Partial fractions

Integral calculus

When expressing an algebraic fraction as partial fractions (as in part **b** of the previous example) the procedure, and the expression we use, depends on the nature of the initial fraction.

- If the fraction is improper, i.e. if the order of the numerator is equal to or greater than the order of the denominator, rearrange the fraction. (As in part **a** of the previous example.)

Now that the only fractions are 'proper' consider the nature of the denominator:

- Denominator has linear factors.

  For example $\dfrac{4x-3}{(x+3)(2x+1)}$. Use $\dfrac{A}{(x+3)}+\dfrac{B}{(2x+1)}$.

- Denominator with a quadratic factor (that does not factorise).

  For example $\dfrac{7x^2-2x+5}{(x-1)(x^2+1)}$. Use $\dfrac{A}{(x-1)}+\dfrac{Bx+C}{(x^2+1)}$.

- Denominator with a repeated linear factor.

  For example $\dfrac{2(3x^2+3x-10)}{(x+3)(x-1)^2}$. Use $\dfrac{A}{(x+3)}+\dfrac{B}{(x-1)}+\dfrac{C}{(x-1)^2}$.

[We do not need a $\dfrac{Cx+D}{(x-1)^2}$ term because $\dfrac{Cx+D}{(x-1)^2}=\dfrac{C(x-1)+C+D}{(x-1)^2}$

$$=\frac{C}{(x-1)}+\frac{C+D}{(x-1)^2}]$$

Some calculators can express fractions in terms of partial fractions, as the display on the right suggests.

$$\text{expand}\left(\frac{4x-3}{(x+3)(2x+1)}, x\right)$$
$$\frac{-2}{(2\cdot x+1)}+\frac{3}{(x+3)}$$
$$\text{expand}\left(\frac{7x^2-2x+5}{(x-1)(x^2+1)}, x\right)$$
$$\frac{2\cdot x}{x^2+1}+\frac{5}{x-1}$$
$$\text{expand}\left(\frac{2(3x^2+3x-10)}{(x+3)(x-1)^2}, x\right)$$
$$\frac{1}{x+3}+\frac{5}{x-1}-\frac{2}{(x-1)^2}$$

Try to obtain these same results yourself, algebraically, using the methods outlined above.

Thus
$$\int\frac{4x-3}{(x+3)(2x+1)}\,dx=-\ln|2x+1|+3\ln|x+3|+c.$$
$$\int\frac{7x^2-2x+5}{(x-1)(x^2+1)}\,dx=\ln(x^2+1)+5\ln|x-1|+c.$$
$$\int\frac{2(3x^2+3x-10)}{(x+3)(x-1)^2}\,dx=\ln|x+3|+5\ln|x-1|+\frac{2}{(x-1)}+c.$$

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Exercise 9E

Determine the following integrals.

1 $\int \frac{7}{x}\,dx$

2 $\int \left(3x^2 - \frac{4}{x}\right)dx$

3 $\int \frac{8x}{x^2+6}\,dx$

4 $\int \tan 2x\,dx$

5 $\int \frac{x+2}{x}\,dx$

6 $\int \frac{x}{x+2}\,dx$

7 $\int \frac{2x-3}{x}\,dx$

8 $\int \frac{x}{2x-3}\,dx$

9 $\int \frac{x^2+4x+1}{x+3}\,dx$

10 $\int \frac{5x+3}{x(x+1)}\,dx$

11 $\int \frac{4x-7}{(x+2)(x-3)}\,dx$

12 $\int \frac{5x^2-2x+18}{(x-1)(x^2+6)}\,dx$

13 $\int \frac{7x^2+8x-4}{(x+1)(x^2+x-1)}\,dx$

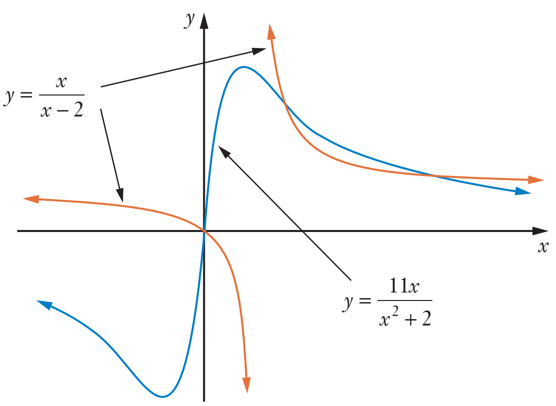
14 $\int \frac{5x^2-10x-3}{(x+1)(x-1)^2}\,dx$

15 $\int \frac{8x^2-44x+25}{(2x+1)(x-3)^2}\,dx$

16 The graph on the right shows

$y = \frac{x}{x-2}$ and $y = \frac{11x}{x^2+2}$.

Prove algebraically that the small region enclosed by these curves has an area of $\left(-5 + \frac{7}{2}\ln 6\right)$ square units.



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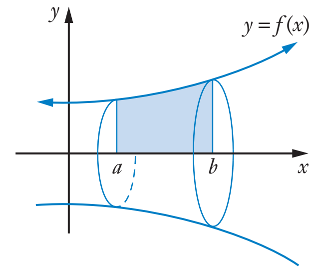
Solids of revolution

Volumes of revolution

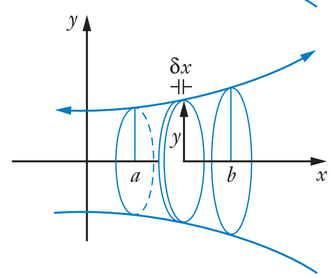
Suppose that we take the area under $y = f(x)$, from $x = a$ to $x = b$, and rotate it about the x-axis one complete revolution (see diagram).



How could we determine the volume of the solid so formed?

The answer is to approach the problem in the same way as we did when considering area: divide the shape up into a large number of pieces each of thickness δx.

One such piece is shown in the diagram on the right.



This will be, approximately, a circular disc of thickness δx and radius y. Hence its volume $\approx \pi y^2\ \delta x$.

$$\text{Thus, total volume} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2\ \delta x$$

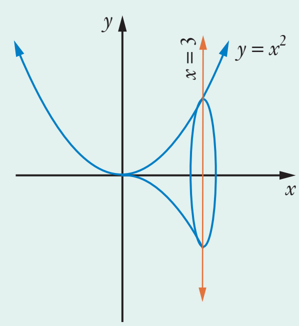
$$= \int_a^b \pi y^2\ dx$$

EXAMPLE 11

Find the volume of the solid formed when the area enclosed by the curve $y = x^2$, the x-axis and the line $x = 3$ is rotated through one revolution about the x-axis.

Solution

The solid involved can be seen in the diagram:



$$\text{Required volume} = \int_0^3 \pi y^2\ dx$$

$$= \int_0^3 \pi (x^2)^2\ dx$$

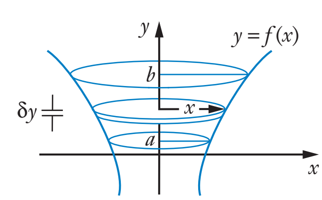
$$= \int_0^3 \pi x^4\ dx$$

$$= \frac{243\pi}{5} \text{ units}^3$$

(Does your calculator have any built-in routines for such a calculation? Investigate.)

Rotation about the y-axis

To determine the volume of an object formed by rotating an area made with the y-axis, about the y-axis, we again consider a small circular disc, this time of thickness δy and radius x, see diagram.



$$\text{Required volume} = \lim_{\delta y \to 0} \sum_{y=a}^{y=b} \pi x^2\ \delta y$$

$$= \int_a^b \pi x^2\ dy$$

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Exercise 9F

For some of the questions in this exercise, evaluate the definite integrals using your calculator and for others show full algebraic reasoning to determine exact answers.

1 Find the volume of the solid formed when the area enclosed by $y = x^2$, the x-axis and the line $x = 2$ is rotated through one revolution about the x-axis.

2 Find the volume of the solid formed when the area enclosed by $y = 3x^2$, the x-axis and the line $x = 1$ is rotated through one revolution about the x-axis.

3 Find the volume of the solid formed when the area between the curve $y = \sqrt{x}$ and the x-axis from $x = 1$ to $x = 4$ is rotated through one revolution about the x-axis.

4 Find the volume of the solid formed when the area enclosed by the x-axis, the straight line $y = 2x + 1$ and the lines $x = 2$ and $x = 3$ is rotated through one revolution about the x-axis.

5 Find the volume of the solid formed by rotating about the x-axis through one revolution, the area between the curve $y = \dfrac{1}{x}$ and the x-axis from

a $x = 1$ to $x = 2$, **b** $x = 2$ to $x = 3$.

6 Find the volume of the solid formed when the area between $y = x^2 + 1$ and the x-axis from $x = -1$ to $x = 2$ is rotated through one revolution about the x-axis.

7 Use integration to determine the volume of the cone formed by rotating the area enclosed by the line $y = 0.5x$, the x-axis and the line $x = 6$ through one revolution about the x-axis.

Show that your answer is consistent with the volume of a right cone of perpendicular height h and base radius r being $\dfrac{\pi r^2 h}{3}$.

8 Find the volume of the solid formed when the area enclosed by $y = \sqrt{\sin x}$ and the x-axis for $0 \le x \le \pi$ is rotated through one revolution about the x-axis.

9 Find the volume of the solid formed when the area enclosed by $y = \sin x$, and the x-axis for $0 \le x \le \pi$ is rotated through one revolution about the x-axis.

10 Find the volume of the solid formed by rotating the area enclosed between $y = x^2$ and $y = x$ through one revolution about the x-axis.

11 Find the volume of the solid formed by rotating the area enclosed between the curves $y = 0.125x^2$ and $y = \sqrt{x}$ through one revolution about the x-axis.

12 Find the volume of the solid formed when the area enclosed between the curves

$$y = 3\cos x \text{ and } y = \cos x, \text{ from } x = -\frac{\pi}{2} \text{ to } x = \frac{\pi}{2},$$

is rotated through one revolution about the x-axis.

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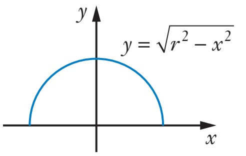
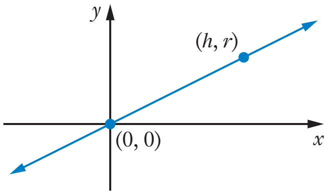
13 By considering the rotation of the area enclosed between $y = \sqrt{r^2 - x^2}$ and the x-axis use calculus to determine the formula for the volume of a sphere of radius r.



14 The diagram on the right shows a straight line passing through (0, 0) and (h, r). By considering the rotation of the area between this line and the x-axis for $0 \le x \le h$ determine the formula for the volume of a right cone of perpendicular height h and base radius r.

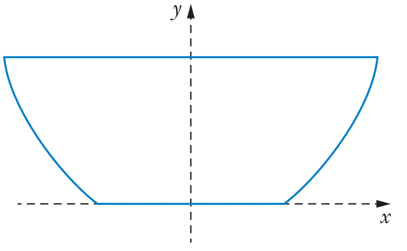


15 Find the volume of the solid formed when the area lying in the first quadrant and enclosed by $y = x^2$, the line $y = 2$ and the y-axis, is rotated through one revolution about the y-axis.

16 Use calculus to determine the exact volume of the solid formed when the area between $y = x\sqrt{5}$ and the y-axis, from $y = 1$ to $y = 2$, is rotated through 360° about the y-axis.

Check your answer using the fact that a right cone of perpendicular height h and base radius r has a volume given by $\dfrac{\pi r^2 h}{3}$.

17 The diagram on the right shows the cross-section of a bowl made by rotating about the y-axis that part of the curve $y = x^2 - 3$ that lies between the x-axis and the line $y = 12$. Neglecting the thickness of the material determine the capacity of the bowl if on each axis 1 unit represents 1 centimetre.



18 A machine component is modelled on computer by rotating for one revolution about the x-axis the area in the first quadrant enclosed by the x-axis for $0 \le x \le 1$, $y = \sqrt{x}$, $y = \sqrt{x - 1}$ and $x = 4$. Determine the volume of the solid so formed.

19 The team designing the nose cone of a small space probe are considering two possibilities.

In one possibility the area between $y = \dfrac{\sin x}{2}$ and the x-axis, from $x = 0$ to $x = \dfrac{\pi}{2}$, is rotated one revolution about the x-axis to give the desired shape.

The other possibility rotates the area between $y = \sqrt{\dfrac{x}{2\pi}}$ and the x-axis from $x = 0$ to $x = \dfrac{\pi}{2}$ one revolution about the x-axis to give the desired shape.

If one unit on each axis is 1 metre determine the *exact* volume of each design.

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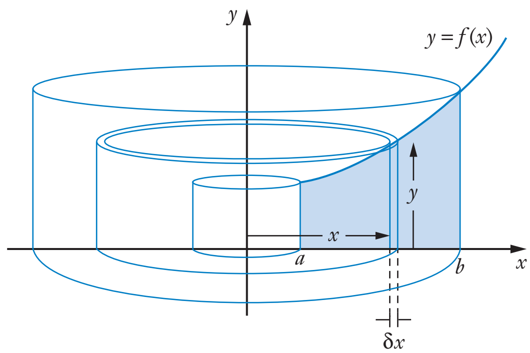
20 A 'paraboloid' is formed by revolving a parabola, $y = kx^2$, about its axis of symmetry. The paraboloid is bounded by a plane cutting the axis of symmetry perpendicularly at the point (0, 20). The intersection of this plane and the paraboloid is a circle of radius 4 units. Determine the volume of the paraboloid.

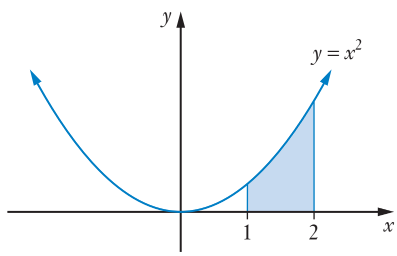
21 Area made with the *x*-axis, rotated about the *y*-axis

By considering the rotation of a small rectangle of thickness δx, height y and located a distance x from the y-axis, find a formula involving a definite integral for the volume of the solid formed by rotating the shaded area shown in the diagram one revolution about the y-axis.

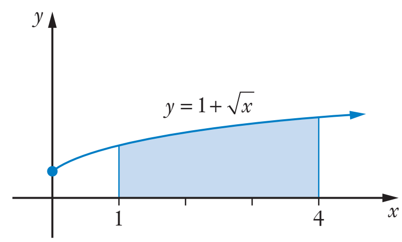


Hence determine the volume of the solids obtained by rotating each of the following shaded areas about the y-axis.

a

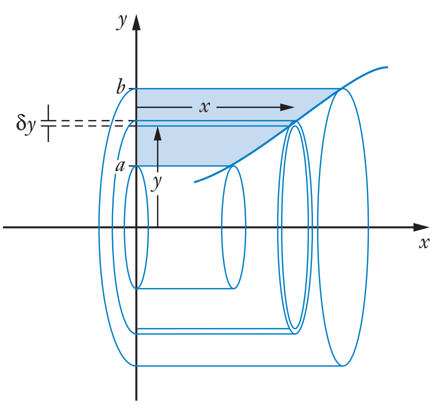


b



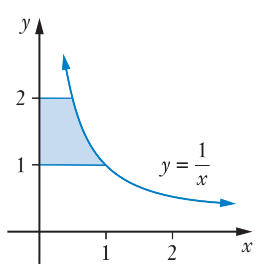
22 Area made with the *y*-axis, rotated about the *x*-axis

By considering the rotation of a small rectangle of thickness δy, length x and located a distance y from the x-axis, find a formula involving a definite integral for the volume of the solid formed by rotating the shaded area shown in the diagram one revolution about the x-axis.

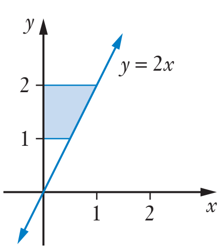


Hence determine the volume of the solids obtained by rotating each of the following shaded areas about the x-axis.

a



b



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Numerical integration

Although, at this stage of the course, we are now able to integrate many functions using suitable substitutions, partial fractions or just with our knowledge of antidifferentiation, it would be quite wrong to assume that we are now able to algebraically integrate any function we might be given. Consider, for example,

$$\int \ln x \, dx, \qquad \int e^x \sin x \, dx, \qquad \int e^{x^2} \, dx.$$

The first two of these can in fact be reasonably easily integrated using a method called *integration by parts*, which features as an extension activity at the end of this chapter. However not all functions can be algebraically integrated and the third example above is one such case. Faced with the task of determining a definte integral for such a function, we can fall back on the basic definition of integration as the limit of a sum and obtain an approximate answer *numerically*. Hence the title of this section, **numerical integration**.

Asked to evaluate $\int_0^2 e^{x^2} \, dx$, for example, our calculator uses such a numerical method to determine an answer.

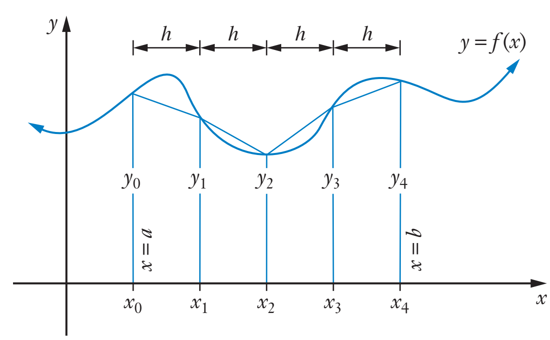
$\int_0^2 e^{x^2} dx$

16.45262777

Suppose we want to determine $\int_a^b f(x)\, dx$, and $f(x)$ is not easily integrated algebraically.

We could divide the area under $y = f(x)$, from $x = a$ to $x = b$, into a number of equal width trapezoidal strips and sum the areas.

The diagram on the right shows 4 such strips.



$$\begin{aligned} \text{Area} &= h\left[\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + \frac{y_3 + y_4}{2}\right] \\ &= \frac{1}{2}h[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4] \end{aligned}$$

Trapezoidal rule

For the general case, with n strips, this trapezoidal approach gives the **trapezium rule**, or **trapezoidal rule**:

$$\int_a^b f(x)\, dx \approx \frac{1}{2}h[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n]$$

where $\qquad y_0 = f(x_0) = f(a), \qquad y_n = f(x_n) = f(b) \qquad$ and $\qquad h = \dfrac{b-a}{n}$.

Another method is to model the top of each strip as being parabolic in shape. Using an even number of such strips this gives **Simpson's rule**:

$$\int_a^b f(x)\, dx \approx \frac{1}{3}h[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n]$$

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Applying the trapezium rule, with $n = 4$, to estimate $\int_0^2 e^{x^2}\,dx$:

$$\int_0^2 e^{x^2}\,dx \approx \frac{1}{2} \times 0.5 \times [y_0 + 2y_1 + 2y_2 + 2y_3 + y_4] \quad \text{with } y_0 = e^{0^2},\ y_1 = e^{0.5^2},$$
$$\approx 20.64 \qquad y_2 = e^{1^2},\ y_3 = e^{1.5^2},\ y_4 = e^{2^2}.$$

Using Simpson's rule:

$$\int_0^2 e^{x^2}\,dx \approx \frac{1}{3} \times 0.5 \times [y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$
$$\approx 17.35$$

Neither of the above estimates are particularly close to the calculator value given on the previous page, but then we have only used four strips. If instead we considered twenty strips, and used a computer spreadsheet to assist us:

	A	B	C	D	E	F	G	H	I
1					*x*	*y*			
2	a	0		0	0	1			
3	b	2		1	0.1	1.01005017			
4	n	20		2	0.2	1.04081077			
5	h	0.1		3	0.3	1.09417428			
6				4	0.4	1.17351087			
7				5	0.5	1.28402542			
8				6	0.6	1.43332941			
9				7	0.7	1.63231622			
10				8	0.8	1.89648088			
11				9	0.9	2.24790799			
12				10	1	2.71828183			
13				11	1.1	3.35348465			
14				12	1.2	4.22069582			
15				13	1.3	5.41948071			
16				14	1.4	7.09932707			
17				15	1.5	9.48773584			
18				16	1.6	12.9358173			
19				17	1.7	17.9933096			
20				18	1.8	25.5337217			
21				19	1.9	36.9660528		**By Trapezium Rule**	**16.6339588**
22				20	2	54.59815		**By Simpson's Rule**	**16.4552084**

Use a computer spreadsheet to estimate definite integrals, using the trapezium rule and Simpson's rule, for other functions. Then compare your approximation to that given by your calculator.

Are there any online calculators for the trapezium rule (trapezoidal rule) and Simpson's rule? Investigate.

The trapezium rule and Simpson's rule are not the only rules available for numerical integration. Do some research on the internet to investigate the midpoint rule, the Newton-Cotes rules and others.

Extension: Integration by parts

According to the product rule $\quad \dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$.

Integrating with respect to x: $\quad uv = \int v\dfrac{du}{dx}\,dx + \int u\dfrac{dv}{dx}\,dx$

Rearranging gives

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$$

This is the formula for *integration by parts*.

EXAMPLE

Use integration by parts to determine $\int xe^x\,dx$.

Solution

Let $\quad u = x \quad$ and $\quad \dfrac{dv}{dx} = e^x$

then $\quad \dfrac{du}{dx} = 1 \quad$ and $\quad v = e^x$

Using $\quad \int u\dfrac{dv}{dx}\,dx = uv - \int v\dfrac{du}{dx}\,dx$

$$\therefore \quad \int xe^x\,dx = xe^x - \int e^x(1)\,dx$$
$$= xe^x - e^x + c$$

Note that when determining v from $\dfrac{dv}{dx}$ we said $v = e^x$. What about the constant? Would our answer have been different had we used $v = e^x + k$?

Exercise

Use integration by parts to determine each of the following integrals.

1 $\int x\sin x\,dx$ **2** $\int x\cos x\,dx$ **3** $\int 3x\sin 2x\,dx$ **4** $\int xe^{2x}\,dx$

5 $\int x^2\ln x\,dx$ **6** $\int x(x+2)^5\,dx$ **7** $\int x\sqrt{2x+1}\,dx$ **8** $\int x^2e^x\,dx$

9 $\int x^2\sin x\,dx$ **10** $\int 2x^3e^{x^2}\,dx$

Now try the following 'sneaky' ones, again using integration by parts.

11 $\int \ln x\,dx$ (Yes it can be done by parts.) **12** $\int e^x\sin x\,dx$ **13** $\int e^x\cos 2x\,dx$

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Miscellaneous exercise nine

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters in this unit, and the ideas mentioned in the Preliminary work section at the beginning of this unit.

For each of questions **1** to **8** find an expression for $\frac{dy}{dx}$.

1 $y = (2x+1)^3$

2 $y = 4\cos 3x + 3\sin 4x$

3 $y = \frac{\sin^4 x}{x}$

4 $y = \frac{1+2\sin x}{1+\cos x}$

5 $y = \frac{\sin 2x}{1+\sin 2x}$

6 $5xy + 2y^3 = 3x^2 - 7$

7 $x = 3t^2 - 5t,\ y = 3 - 4t^3$

8 $x\cos y = y\sin x$

9 Find the constants a and b given that for $\{x \in \mathbb{R}: x \neq \pm 1\}$

$$\frac{a}{x-1} + \frac{b}{x+1} = \frac{7x-5}{x^2-1}.$$

Hence find an expression for $\int \frac{7x-5}{x^2-1}\,dx$.

10 Determine each of the following indefinite integrals.

a $\int 4\cos 8x\,dx$

b $\int 2x(3+x^2)^5\,dx$

c $\int (2-3x)\sqrt[3]{x+3}\,dx$

d $\int \sin^5 2x\cos 2x\,dx$

e $\int \sin^2 \frac{x}{2}\,dx$

f $\int \cos^3 \frac{x}{2}\,dx$

g $\int \sin^3 2x\,dx$

h $\int 6\sin 2x\cos x\,dx$

i $\int 6\cos 2x\sin x\,dx$

11 Find the equation of the tangent to

a $x^2 + xy = 1 + y^2$ at the point (2, 3).

b $x^3 + y^3 = 35$ at the point (2, 3).

12 Find the volume of the solid formed by rotating the area enclosed between the curve $y = \frac{1}{\sqrt{x}}$ and the x-axis, from $x = 1$ to $x = 4$, through one revolution about the x-axis.

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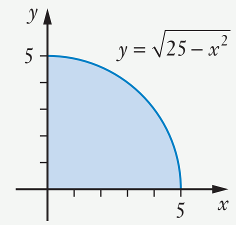
13 The length of a particular rectangle is four times its width and this ratio is maintained as the width is increased at 2 mm/s. Find the rate of increase in the area of the rectangle when the width is 15 cm.

14 The area of the quarter circle shown shaded on the right is given by $\int_0^5 \sqrt{25 - x^2}\, dx$.

Use the substitution $x = 5 \sin u$ to evaluate this definite integral exactly and show that your answer is consistent with the area of a circle of radius r being πr^2.

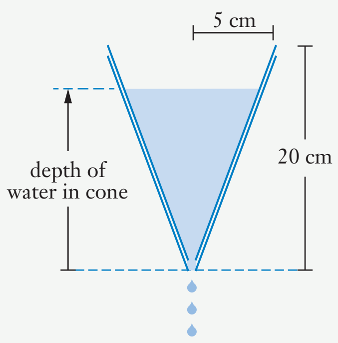


15 Showing full algebraic reasoning, determine the following definite integral giving your answer as an exact value.

$$\int_1^2 \frac{3x^2 + 5x - 1}{(x + 2)(x + 1)^2}\, dx$$

16 The diagram shows a funnel in the shape of an upturned cone of height 20 cm and 'base' radius 5 cm.

If water flows out of the funnel at 5 cm^3/s, how fast is the water level falling at the instant that the water in the cone has a depth of 10 cm?



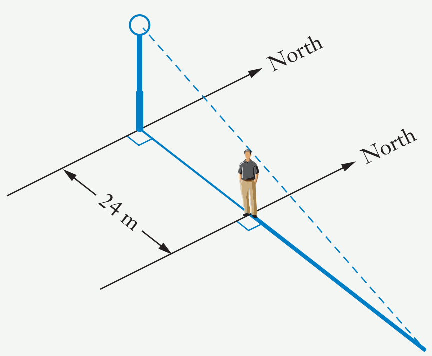
17 The diagram shows a person of height 1.95 m standing 24 metres due east of a lamp post that holds a light that is 4.2 m above the ground.

If the person runs at 5 m/s, find how fast the length of the person's shadow is changing 2 seconds later if the direction in which the person runs is

a due east,

b due west,

c due north.



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